Mapping of the B_N-type Calogero-Sutherland-Moser system to decoupled Harmonic Oscillators

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Abstract

The B_N-type Calogero-Sutherland-Moser system in one-dimension is shown to be equivalent to a set of decoupled oscillators by a similarity transformation. This result is used to show the connection of the A_N and B_N type models and explain the degeneracy structure of the later. We identify the commuting constants of motion and the generators of a *linear* W_∞ algebra associated with the B_N system.

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Typeset using REVTEX

In the recent literature, there is an increasing interest in the quantum mechanical, exactly solvable N-particle systems in one-dimension [1-4]. All these models are characterised by non-trivial, long-range interactions between particles and Jastrow-type correlated many-body wave functions. Interestingly, they have been found relevant for the description of universal properties of various physical systems. These include spin chains [5], quantum Hall effect [6], universal conductance fluctuations in mesoscopic systems [7], two-dimensional gravity [8], gauge theories [9] and random matrices [2,10].

In a recent paper [11], the present authors have constructed a similarity transformation which maps the A_N-type Calogero-Sutherland (CS) model [1,2], having pair-wise inverse-square and harmonic interactions, to decoupled harmonic oscillators. Starting from the symmetrised eigenfunctions of N free harmonic oscillators, the wave functions of the CS model can then be constructed explicitly. In general, the Hamiltonians which can be brought through a suitable transformation to the generalised form $\tilde{H} = \sum_i x_i \frac{\partial}{\partial x_i} + c + \hat{F}$ can also be mapped to $\sum_i x_i \frac{\partial}{\partial x_i} + c$ by a similarity transformation. The operator that accomplishes this is given by $\exp\{-d^{-1}\hat{F}\}$; where, $\hat{F}$ is any homogeneous function of $\frac{\partial}{\partial x_i}$ and x_i with degree d, and c is a constant. For the normalizability of the wave functions, one needs to check that the action of $\exp\{-d^{-1}\hat{F}\}$ on an appropriate linear combination of the eigenstates of $\sum_i x_i \frac{\partial}{\partial x_i}$ yields a polynomial solution.

In this letter, the B_N-type Calogero-Sutherland-Moser (CSM) model [4] is diagonalised using the above procedure. We (i) prove the equivalence of B_N-type model to decoupled oscillators, (ii) construct the complete set of eigenfunctions for this model from those of harmonic oscillators, (iii) show the existence of N independent commuting constants of motion and a *linear* W_∞ algebra. We also point out the connection of A_N and B_N-type models and explain the origin of degeneracies.

The CSM Hamiltonian (in the units $\hbar = m = \omega = 1$) is given by

$$H = -\frac{1}{2}\sum_{i=1}^{N}\partial_i^2 + \frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{1}{2}g^2 \sum_{\substack{i,j=1\\ i\neq j}}^{N} \left\{ \frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right\} + \frac{1}{2}g_1^2 \sum_{i=1}^{N} \frac{1}{x_i^2} \quad , \qquad (1)$$

where, $\partial_i \equiv \partial/\partial x_i$; g^2 and g_1^2 are coupling constants. Notice that, the above Hamiltonian is

not translational invariant unlike the A_N-type model. Since, the ground-state wave function of H, when the system is quantised as bosons, is given by

$$\psi_0 = \prod_{1\leq j<k\leq N} |x_i - x_j|^{\lambda} |x_i + x_j|^{\lambda} \prod_{k}^{N} |x_k|^{\lambda_1} \exp\{-\frac{1}{2}\sum_i x_i^2\} \qquad , \tag{2}$$

one can make the following similarity transformation on the Hamiltonian

$$\tilde{H} \equiv \psi_0^{-1} H \psi_0 = \sum_i x_i \partial_i + E_0 + \hat{F} \qquad . \tag{3}$$

Here,

$$\hat{F} \equiv -\left(\frac{1}{2}\sum_i \partial_i^2 + \lambda \sum_{i<j} \frac{1}{(x_i^2 - x_j^2)}(x_i\partial_i - x_j\partial_j) + \lambda_1 \sum_i \frac{1}{x_i}\partial_i\right) \qquad ,$$

$g^2 = \lambda(\lambda - 1)$, $g_1^2 = \lambda_1(\lambda_1 - 1)$ and $E_0 = N(\frac{1}{2} + (N-1)\lambda + \lambda_1)$ is the ground-state energy. The eigenfunctions of $\tilde{H}$ must be totally symmetric with respect to the exchange of any two particle coordinates; the bosonic or the fermionic nature of the wave function being contained in ψ_0. One can easily establish the following commutation relation;

$$[\sum_i x_i \partial_i \ , \ \exp\{\hat{F}/2\}] = -\hat{F} \exp\{\hat{F}/2\} \qquad . \tag{4}$$

Making use of (4) in (3), one gets

$$\exp\{-\hat{F}/2\} \tilde{H} \exp\{\hat{F}/2\} = \sum_i x_i \partial_i + E_0 \qquad . \tag{5}$$

Hence, the similarity transformation by the operator $\hat{S} \equiv \psi_0 \exp\{\hat{F}/2\}$ diagonalises H *i.e.*, $\hat{S}^{-1} H \hat{S} = \sum_i x_i \partial_i + E_0$. Furthermore, the following similarity transformation on (5) makes the connection of B_N-type model with the decoupled oscillators explicit:

$$G\hat{E}\hat{S}^{-1} H \hat{S} \hat{E}^{-1} G^{-1} = -\frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_i x_i^2 + (E_0 - \frac{1}{2}N) \qquad , \tag{6}$$

where, $\hat{E} \equiv \exp\{-\frac{1}{4}\sum_i \frac{\partial^2}{\partial x_i^2}\}$ and $G \equiv \exp\{-\frac{1}{2}\sum_i x_i^2\}$. This is one of the main results of this letter.

At this stage, it is interesting to note that, (5) can also be made equivalent to the Calogero-Sutherland model. This is achieved by first defining an operator $\hat{T} \equiv ZG\exp\{-\hat{A}/2\}$, where $Z \equiv \prod_{i<j} |x_i - x_j|^{\alpha}$ and $\hat{A} \equiv \frac{1}{2}\sum_i \partial_i^2 + \alpha \sum_{i<j} \frac{1}{(x_i - x_j)}(\partial_i - \partial_j)$:

$$\hat{T}\left(\sum_i x_i\partial_i + E_0\right)\hat{T}^{-1} = -\frac{1}{2}\sum_i \partial_i^2 + \frac{1}{2}\sum_i x_i^2 + \sum_{i<j}\frac{\alpha(\alpha-1)}{(x_i-x_j)^2} + (E_0 - E_0') \qquad , \tag{7}$$

here, $E_0' = \frac{N}{2}[1+\alpha(N-1)]$ is the ground-state energy of the CS model. In other words, the operator $\hat{S}\hat{T}^{-1}$ maps CSM to CS model.

From (5), one can also obtain the creation and annihilation operators as $a_i^+ = \hat{S}x_i\hat{S}^{-1}$ and $a_i^- = \hat{S}\partial_i\hat{S}^{-1}$: the symmetrised form of the operators $K_i^+ = \frac{1}{2}a_i^{+2}$ acts on the ground-state obtained from $a_i^-|0>=0; i=1,2,\cdots,N$ and creates the eigenstates of this B_N-type model. In terms of the creation and annihilation operators, the CSM Hamiltonian can be written as $H = \sum_i H_i + (E_0 - N/2)$, where $H_i \equiv a_i^+a_i^- + \frac{1}{2}$. It can be easily checked that, the commutation relations $[K_i^-\ ,\ K_j^+] = \delta_{ij}H_i$ and $[H_i\ ,\ K_j^\pm] = \pm 2\delta_{ij}K_i^\pm$ generate N copies of $SU(1,1)$ algebra; here, $K_i^- = \frac{1}{2}a_i^{-2}$. It is worth noticing here that these, N, commuting $SU(1,1)$ generators act on the even sector of the harmonic oscillator basis and hence generate a complete set of eigenfunctions. In this respect, it should be noted that, the eigenfunctions of the CS model is made up of both the even and odd sectors of the oscillator basis [11].

One can also define $<<0|S_n(\{\frac{1}{2}a_i^{-2}\}) =<<n|$ and $S_n(\{\frac{1}{2}a_i^{+2}\})|0>=|n>$ as the bra and ket vectors; S_n is a symmetric homogeneous function of degree n and $<<0|\frac{1}{2}a_i^{+2} = \frac{1}{2}a_i^{-2}|0>=0$. Since all the N $SU(1,1)$ algebras are decoupled, the inner product between these bra and ket vectors proves that any ket $|n>$, with a given partition of n, is orthogonal to all the bra vectors, with different n and also to those with different partitions of the same n. The normalisation for any state $|n>$ can also be found out from the ground state normalisation, which is known [13].

We can also make use of (5) for the explicit construction of the eigenfunctions of (1). One notices that an arbitrary homogeneous function is an eigenfunction of $\sum_i x_i\partial_i$; however, only the homogeneous symmetric functions of the square of the particle co-ordinates are the ones, on which the action of $\exp\{\hat{F}/2\}$ gives a polynomial solution. There are several related basis sets available for these functions, viz., Schur functions, monomial symmetric functions, complete symmetric functions and Jack polynomials[12]. Jack polynomials provide one

special choice for this basis set which has been studied recently by T.H. Baker and P.J. Forrester [13]. It is also worth noting that if $\lambda = 1$, then (5) reduces to the differential equation, in $y_i = x_i^2$ variables for the multivartiate Lagurre polynomials [14]. This is the basic reason why the action of the operator $\exp\{\hat{F}/2\}$ on the homogeneous symmetric functions of the square of the particle co-ordinates gives a polynomial solution. For the sake of illustration, we choose here the power sum basis $P_l \equiv \sum_i (x_i^2)^l$. The eigenfunctions and the energy eigenvalues are respectively given by

$$\psi_n = \psi_0 \left[\exp\{\hat{F}/2\} \prod_{l=1}^{N} P_l^{n_l} \right] \qquad , \tag{8}$$

and $E_n = 2\sum_l^N l n_l + E_0$; $n = \sum_l l n_l$. From this it is clear that, $\hat{S}\hat{T}^{-1}$ maps all eigenfunctions of H to the even sector of CS model (or to that of harmonic oscillators, if $\alpha = 0$ or 1).

The quantum integrability of the CSM model and the identification of the constants of motion become transparent after establishing its equivalence to free oscillators. It is easy to verify that $[H \ , \ H_k] = [H_i \ , \ H_j] = 0$ for $i, j, k = 1, 2, \cdots, N$. Therefore, the set $\{H_1, H_2, \cdots, H_N\}$ provides the N conserved quantities. One can construct linearly independent symmetric conserved quantities from the elementary symmetric polynomials, since they form a complete set:

$$I_1 = \sum_{1 \le i \le N} H_i \ , I_2 = \sum_{1 \le i < j \le N} H_i H_j \ , I_3 = \sum_{1 \le i < j < k \le N} H_i H_j H_k, \cdots, I_N = \prod_{i=1}^{N} H_i. \tag{9}$$

Here, we would like to point out that, the present proof is entirely different from earlier works [15].

With the generators of the above mentioned $SU(1,1)$ algebras of CSM, one can define a linear W_∞ algebra for which there exist several basis sets [16].

In conclusion, we mapped the B_N-type Calogero-Sutherland-Moser model to decoupled oscillators and algebraically constructed the complete set of eigenfunctions, including the degenerate ones, from the symmetrised eigenstates of the harmonic oscillators. We also showed the existence of N independent commuting constants of motion, a *linear* W_∞ algebra for this system and the connection it has with A_N-type model. The application of this

method to other one and higher-dimensional [17] models can also be carried out in an analogous manner.

The authors are grateful to Profs. V. Srinivasan and S. Chaturvedi for many illuminating discussions. P.K.P would like to acknowledge useful discussions with Prof. A. Khare and Dr. M. Sivakumar. N.G thanks E. Harikumar for useful comments and U.G.C (India) for financial support through the S.R.F scheme.

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